Dynamics of curved domain boundaries in convection patterns

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(Received 24 August 1993; revised manuscript received 14 February 1994)

Curved domain boundaries (DB's) between locally stable convection patterns are studied near the onset of convection, within the framework of the Newell-Whitehead-Segel theory [J. Fluid Mech. **38**, 279 (1969); **38**, 203 (1969)]. We consider the case where there exists a Lyapunov functional. By means of asymptotic methods, the equations of motion for DB's are derived, and their solutions are obtained. It is shown that the behavior of a DB depends strongly on the difference between Lyapunov functional's densities of the coexisting patterns. In the case of a nonzero difference, the normal velocity depends on the orientation of the DB, and caustics can be produced in a finite time. In the case of zero difference, the normal velocity depends on both orientation and distortion of the DB, and the DB tends typically to straighten after a long time.

PACS number(s): 47.20.Ky, 47.27.Te

I. INTRODUCTION

The convection patterns arising spontaneously from a random perturbation in a horizontal layer of fluid heated from below, contain different types of defects. The investigation of the defects dynamics was started in the work of Siggia and Zippelius [1] and continues in a large number of papers (for review, see Ref. [2]). Among them, one can observe linear defects whose existence is a result of the multistability of convection patterns. For instance, if the roll patterns are stable, rolls of different orientations can arise in different spatial domains. These domain boundaries (DB's) are separated by intermediate zones where rectangular [3–5] or hexagonal [5] patterns are sandwiched. Another type of DB is expected in the parameter region where both roll and hexagonal patterns are stable [6]. They separate the domains of patterns with different spatial symmetries.

Pomeau [7] showed that the dynamics of DB in nonequilibrium patterns near the instability threshold can be described by amplitude equations [8,9]

$$\frac{\partial a_j}{\partial t}=-\frac{\delta \mathcal{F}}{\delta a_j^*}\ , \tag{1.1}$$

where $j=1,\dots,N$ (in every equation in which j appears and is not summed over, the equation is valid for each value of j), with a certain Lyapunov functional $\mathcal{F}(a_j,\dots,a_N)$. On the basis of these equations, the plane DB's in convection patterns were considered in Ref. [5]. These plane DB's move with a constant velocity in the direction of the patterns with higher density of Lyapunov functional (LF), and do not move if the densities of LF are equal. It turns out that the velocity of a DB and its contribution to the LF depend on its orientation. The experimental observations [10] confirm qualitatively these theoretical predictions.

In the present paper, DB's of curved shape are studied.

II. FORMULATION OF THE PROBLEM

We consider the convection in an infinite horizontal layer heated from below when the Rayleigh number R is close to the critical value R_c $[R=R_c(1+\delta^2),\delta\ll 1]$. We use the Newell-Whitehead-Segel [8,9] approach. Disregarding the possibility of *zigzag instability* of solutions and *phase diffusion effects*, we start with the following system of equations for real amplitudes $A_j(X,Y,\tau)$:

$$\frac{\partial A_j}{\partial \tau}-(\mathbf{n}_j\cdot\nabla)^2A_j=-\frac{\partial V(A_1,\dots,A_N)}{\partial A_j}\ , \tag{2.1}$$

where the potential function V is defined by

$$V=-\tfrac{1}{2}\Gamma\sum_{j=1}^{N}A_j^2+\tfrac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N}\beta_{jk}A_j^2A_k^2+\tfrac{1}{3}\sum_{l,m,n}A_lA_mA_n\ , \tag{2.2}$$

and it is assumed that V is bounded from below, $\{\mathbf{n}_j\}$ is a set of unit vectors. The details can be found in Refs. [5,7,8, and 9].

In Eqs. (2.1) and (2.2), the indices l,m,n are such that the corresponding modes constitute a resonantly coupled triad, i.e., $\mathbf{n}_l+\mathbf{n}_m+\mathbf{n}_n=\mathbf{0}$, A_j are the scaled amplitude functions, the parameter Γ is the scaled linear growth rate of the disturbance with the critical instability wave number, the nonlinear interaction coefficients β_{jk} depend on the angle θ_{jk} between $\mathbf{n}_j$ and $\mathbf{n}_k$. The resonant cubic terms should be dropped if the dependence of the viscosity and heat conductivity on the temperature is negligible.

Equation (2.1) can be written also in the form

$$\frac{\partial A_j}{\partial \tau}=-\frac{\delta \mathcal{F}}{\delta A_j}\ , \tag{2.3}$$

where $\mathcal{F}$, the Lyapunov functional, is

$$\mathcal{F}=\int dX\,dY\left\{\tfrac{1}{2}\sum_j[(\mathbf{n}_j\cdot\nabla)^2A_j]^2+V\right\}. \tag{2.4}$$

In real systems, the domains of ordered patterns arise from disordered disturbances induced by noise, after a certain transient ordering process. To simplify the problem, we assume that initial disturbances have a spatial scale of $\epsilon^{-1}\gg 1$. We then expect that large homogeneous domains will arise, separated by domain boundaries with a much smaller width.

Indeed, assuming

$$A_j(X,Y,\tau;\epsilon)=A_j^0(x,y,\tau)+O(\epsilon^2)\ , \tag{2.5}$$

where $x=\epsilon X, y=\epsilon Y$, and substituting (2.5) into (2.1), we get in ϵ^0 order

$$\frac{\partial A_j^0}{\partial\tau}=-\frac{\partial V(A_1^0,\ldots,A_N^0)}{\partial A_j^0}\ , \tag{2.6}$$

hence if $\partial V/\partial A_j^0\neq 0$ for any j, except for extremal points of V, then V will be a monotone function decreasing in time. Therefore, if V has a minimal value, it will tend to that minimum for large τ.

We assume that V has several local minima, every one of these is surrounded by a basin of attraction in which for large τ, the function V tends to its minimum. The "border" between every two such subdomains is a DB: the solution for long time in each of the domains corresponds to a certain convection pattern, and the DB has different patterns on both sides. The dynamics of these DB's is the subject of the present paper.

III. ASYMPTOTIC EXPANSIONS

In order to discuss the dynamics of DB's, we shall use the approach developed by Rubinstein, Sternberg, and Keller [11]. Let us consider two adjacent sub domains with a common DB, situated on the locus

$$\varphi_0(x,y)=0\ . \tag{3.1}$$

The following discussion involves several time scales. The time scale τ is the fastest one, for slower processes $t=\epsilon\tau$, $\eta=\epsilon t$ are defined.

In order to determine the asymptotic expansion of the functions A_j in the DB neighborhood, we assume that the DB moves in such a way that its locus depends on the slow time variables t and η. The DB is situated on $\varphi(x,y,t,\eta)=0$, and its initial position is $\varphi(x,y,0,\eta)=\varphi_0(x,y)$.

We assume that we can expand the functions A_j into asymptotic series of the form

$$A_j(X,Y,t,\epsilon)=a_j^0(z,x,y,\tau,t,\eta)+\epsilon a_j^1(z,x,y,\tau,t,\eta)+O(\epsilon^2)\ , \tag{3.2}$$

where

$$z=\frac{1}{\epsilon}\varphi(x,y,t,\eta)\ . \tag{3.3}$$

By substituting (3.2) into (2.1), we obtain in ϵ^0 order

$$\frac{\partial a_j^0}{\partial\tau}+\frac{\partial\varphi}{\partial t}\frac{\partial a_j^0}{\partial z}=(\mathbf{n}_j\cdot\nabla\varphi)^2\frac{\partial^2 a_j^0}{\partial z^2}-\frac{\partial V(a_1^0,\ldots,a_N^0)}{\partial a_j}\ , \tag{3.4}$$

and in ϵ^1 order

$$\frac{\partial a_j^1}{\partial\tau}+\frac{\partial\varphi}{\partial t}\frac{\partial a_j^1}{\partial z}+\frac{\partial a_j^0}{\partial t}+\frac{\partial\varphi}{\partial\eta}\frac{\partial a_j^0}{\partial z}=(\mathbf{n}_j\cdot\nabla\varphi)^2\frac{\partial^2 a_j^1}{\partial z^2}+[2(\mathbf{n}_j\cdot\nabla\varphi)(\mathbf{n}_j\cdot\nabla)+(\mathbf{n}_j\cdot\nabla)^2\varphi]\times\frac{\partial a_j^0}{\partial z}-\sum_{m=1}^{N}a_m^1\frac{\partial^2 V(a_1^0,\ldots,a_N^0)}{\partial a_m\partial a_j}\ . \tag{3.5}$$

In order to determine the function φ in (3.4), we assume that for large τ the functions a_j^0 tend to stationary solutions which depend on z alone and are independent of τ. Let

$$a_j^0(z,x,y,\tau,t,\eta)\sim Q_j(z,x,y,t,\eta)\quad\text{as }\tau\to\infty\ . \tag{3.6}$$

We substitute (3.6) into (3.4) to obtain

$$-(\mathbf{n}_j\cdot\nabla\varphi)^2Q_j''+\frac{\partial\varphi}{\partial t}Q_j'+\frac{\partial V(Q_1,\ldots,Q_N)}{\partial Q_j}=0\ , \tag{3.7}$$

where a prime means $\partial/\partial z$. This equation can be written also in the form

$$-(\mathbf{n}_j\cdot\mathbf{n}_f)^2\frac{d^2Q_j}{ds^2}+v_n\frac{dQ_j}{ds}+\frac{\partial V(Q_1,\ldots,Q_N)}{\partial Q_j}=0\ , \tag{3.8}$$

where

$$\mathbf{n}_f=\frac{\nabla\varphi}{|\nabla\varphi|}\ ,\quad v_n=\frac{\varphi_t}{|\nabla\varphi|}\ ,\quad s=\frac{z}{|\nabla\varphi|}\ . \tag{3.9}$$

From the matching conditions to the outer solution, we have boundary conditions such that Q_j are different constants far away from both sides of the DB.

IV. THE CASE OF $[V]\neq 0$

From the existence condition of solutions to Eq. (3.7) we obtain the DB *equation of motion*:

$$\frac{\partial\varphi}{\partial t}=-\frac{[V]}{\int_{-\infty}^{\infty}\sum_{j=1}^{N}(Q_j')^2dz}\ , \tag{4.1}$$

where $[V]$ is the potential difference between both sides of the DB in steady state. With the variables (3.9) this equation can be written in the form

$$v_n=-\frac{[V]}{\int_{-\infty}^{\infty}\sum_{j=1}^{N}\left[\frac{dQ_j}{ds}\right]^2ds}\ , \tag{4.2}$$

The eigenvalue v_n depends only on $\mathbf{n}$ so that

$$v_n=v(\mathbf{n}_f)\ . \tag{4.3}$$

In the general case there may be several solutions of (3.8) or none. The existence and uniqueness of a solution for equations of the type of (3.8) was proved by Gardner [12] only for the case $N=2$ (DB with rolls on both sides). We assume here existence and uniqueness of v_n for all N.

The formula (4.2) coincides with one obtained in the case of a plane DB [5], therefore the curved DB moves with the same velocity as a plane DB having the same normal.

We have found that $\partial\varphi/\partial t$ is a function of $\nabla\varphi$, so

$$\frac{\partial\varphi}{\partial t}=F\left[\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial x}\right]\ . \tag{4.4}$$

Solving (4.4) by the method of characteristics, we get the solution

$$\varphi(x,y,t)=\varphi_0(x_0,y_0)+[p_0-q_0F_q(q_0,r_0)-r_0F_r(q_0,r_0)]t\ , \tag{4.5}$$

where

$$y_0=y+F_q(q_0,r_0)t\ ,\quad x_0=x+F_r(q_0,r_0)t\ ,$$
$$q_0=\partial\varphi_0(x_0,y_0)/\partial y\ ,\quad r_0=\partial\varphi_0(x_0,y_0)/\partial x\ ,$$
$$p_0=F(q_0,r_0)\ . \tag{4.6}$$

For the explicit representation,

$$\varphi(x,y,t)=x-f(y,t)\ , \tag{4.7}$$

we have $\nabla\varphi=[1,-(\partial f/\partial y)]$, and we obtain instead of (4.4),

$$\frac{\partial f}{\partial t}=F\left[\frac{\partial f}{\partial y}\right]\ . \tag{4.8}$$

The solution (4.5) is reduced to

$$f(y,t)=f_0(y_0)+[p_0-q_0F_q(q_0)]t\ . \tag{4.9}$$

From (4.9) we obtain the equation:

$$x=f_0(y_0)+[F(f_0'(y_0))-f_0'(y_0)F'(f_0'(y_0))]t\ . \tag{4.10}$$

The time needed for a singularity to occur in the second derivative of the DB shape is

$$t=\frac{1}{F''(f_0'(y_0))f_0''(y_0)}\ . \tag{4.11}$$

Hence, at some points on the DB shape, caustics will appear after a finite interval of time. When this happens, the equations we use are not valid anymore.

V. THE CASE OF $[V]=0$

If $[V]=0$ it follows from the DB equation of motion (4.1) that $\partial\varphi/\partial t=0$, and therefore the DB will not move on the time scale of t. Equation (3.7) becomes

$$-(\mathbf{n}_j\cdot\nabla\varphi)^2Q_j''+\frac{\partial V(Q_1,\ldots,Q_N)}{\partial Q_j}=0\ , \tag{5.1}$$

so that Q_j moves only with the time scale η.

In order to find the motion of the DB on the η time scale, we must proceed to the second order Eq. (3.5). Here we assume that the functions a_j^1 tend to stationary functions of z as τ grows to infinity in the same way as a_j^0. We write

$$a_j^1(z,x,y,\tau,t,\eta)\sim P_j(z,x,y,t,\eta)\ \text{ as } \tau\to\infty\ , \tag{5.2}$$

we substitute a_j^1 of (5.2) into (3.5) using the relation $\partial\varphi/\partial t=0$, and we obtain

$$-(\mathbf{n}_j\cdot\nabla\varphi)^2P_j''+\sum_{m=1}^{N}P_m\frac{\partial^2V(Q_1,\ldots,Q_N)}{\partial Q_m\partial Q_j}=\left[2(\mathbf{n}_j\cdot\nabla\varphi)(\mathbf{n}_j\cdot\nabla)+(\mathbf{n}_j\cdot\nabla)^2\varphi-\frac{\partial\varphi}{\partial\eta}\right]Q_j'\ . \tag{5.3}$$

We calculate the solvability condition of (5.3) using the *Fredholm alternative theorem* and Eq. (5.1), and we obtain the DB motion equation for the case of $[V]=0$:

$$\frac{\partial\varphi}{\partial\eta}=\frac{\sum_{j=1}^{N}[(\mathbf{n}_j\cdot\nabla)^2\varphi+(\mathbf{n}_j\cdot\nabla\varphi)(\mathbf{n}_j\cdot\nabla)]\int_{-\infty}^{\infty}(Q_j')^2dz}{\sum_{j=1}^{N}\int_{-\infty}^{\infty}(Q_j')^2dz}\ . \tag{5.4}$$

With the variables (3.9) this equation can be written in the form

$$v_n=\frac{|\nabla\varphi\sum_{j=1}^{N}(\mathbf{n}_j\cdot\nabla)\left[\frac{(\mathbf{n}_j\cdot\mathbf{n}_f)G_j(\mathbf{n}_j\cdot\mathbf{n}_f)}{|\nabla_\varphi|}\right]}{\sum_{j=1}^{N}G_j(\mathbf{n}_j\cdot\mathbf{n}_f)}\ . \tag{5.5}$$

where

$$G_j(\mathbf{n}_j\cdot\mathbf{n}_f)\equiv\int_{-\infty}^{\infty}\left[\frac{dQ_j}{ds}\right]^2ds\ .$$

This relation resembles the result that was obtained in Ref. [11]. In the case of $[V]=0$ for the reaction-diffusion equation it was found that v_n was proportional to the curvature of the DB.

We consider again the explicit case $\varphi(x,y,t)=x-f(y,\eta)$. It follows from Eq. (5.1) that Q_j depend only on s and on $\partial f/\partial y$. Therefore, Eq. (5.4) gives

$$\frac{\partial f}{\partial\eta}=\frac{\sum_{j=1}^{N}n_{j_y}^2H_j(\omega)-n_{j_x}n_{j_y}\frac{dH_j(\omega)}{d\omega}+n_{j_y}^2\,\omega\frac{dH_j(\omega)}{d\omega}}{\sum_{j=1}^{N}H_j(\omega)}\frac{\partial^2 f}{\partial y^2}, \tag{5.6}$$

where

$$\omega\equiv\frac{\partial f}{\partial y}\ ,\quad\text{and } H_j(\omega)\equiv\int_{-\infty}^{\infty}(Q_j')^2dz\ . \tag{5.7}$$

Thus if we denote the coefficient of $\partial^2f/\partial y^2$ in (5.6) by $K(\omega)$, and differentiate this equation with respect to y, we get

$$\frac{\partial\omega}{\partial\eta}=\frac{\partial}{\partial y}\left[K(\omega)\frac{\partial\omega}{\partial y}\right]\ , \tag{5.8}$$

which resembles the nonlinear diffusion equation with $K(\omega)$ as the diffusion coefficient, which is neither constant nor linear. It is easy to show that if $K(\omega)$ is positive for all ω, then ω tends after a long time to a constant, so that we get a plane DB after a long time. If K is negative for some ω, there may be a singularity in the DB evolution in time.

It should be mentioned that the contribution of the DB to the Lyapunov functional (2.4) can be written in the form of an integral along the DB:

$$\mathcal{F}=\tfrac{1}{2}\int\sum_{j=1}^{N}H_j\left[\frac{\partial f}{\partial y}\right]\left[n_{j_x}-n_{j_y}\frac{\partial f}{\partial y}\right]^2dy\ . \tag{5.9}$$

This implies the relation

$$\frac{\partial f}{\partial \eta}=-\frac{1}{\sum_{j=1}^{N} H_j \left[\frac{\partial f}{\partial y}\right]}\frac{\delta \mathcal{F}}{\delta f}\ , \tag{5.10}$$

which coincides with the inequality $d\mathcal{F}/d\eta \le 0$.

An explicit asymptotic solution for a roll-roll domain boundary is known in the limit $0 \le \beta_{12}-\beta_{11} \ll 1$ [see Ref. [5], Eq. (3.22)]. Using this solution we obtain

$$H_1(\omega) \propto \frac{|n_{1_x}-n_{1_y}\omega|+2|n_{2_x}-n_{2_y}\omega|}{3(|n_{1_x}-n_{1_y}\omega|+|n_{2_x}-n_{2_y}\omega|)^2}\ , \tag{5.11}$$

$$H_2(\omega) \propto \frac{2|n_{1_x}-n_{1_y}\omega|+|n_{2_x}-n_{2_y}\omega|}{3(|n_{1_x}-n_{1_y}\omega|+|n_{2_x}-n_{2_y}\omega|)^2}\ . \tag{5.12}$$

Substitution of (5.11) and (5.12) and their derivatives into $K(\omega)$ gives the result

$$K(\omega)=\frac{2(n_{1_x}n_{2_y}-n_{2_x}n_{1_y})^2}{3(|n_{1_x}-n_{1_y}\omega|+|n_{2_x}-n_{2_y}\omega|)^2} \tag{5.13}$$

which is obviously positive for all ω.

Finally, let us assume that $[V]$ is of order ϵ, i.e.,

$$[V]=\epsilon[V^1]+O(\epsilon^2)\ . \tag{5.14}$$

We denote

$$V(a_1,\ldots,a_N)=V^0(a_1,\ldots,a_N)+\epsilon V^1(a_1,\ldots,a_N)+O(\epsilon^2)\ . \tag{5.15}$$

We repeat the same calculations as for $[V]\neq 0$ and $[V]=0$, substitute $[V^0]=0$, and instead of (5.4) we obtain

$$\frac{\partial \varphi}{\partial \eta}=\frac{-[V^1]+\sum_{j=1}^{N}[(\mathbf{n}_j\cdot\nabla)^2\varphi+(\mathbf{n}_j\cdot\nabla\varphi)(\mathbf{n}_j\cdot\nabla)]\int_{-\infty}^{\infty}(Q_j')^2 dz}{\sum_{j=1}^{N}\int_{-\infty}^{\infty}(Q_j')^2 dz}\ . \tag{5.16}$$

We substitute again the explicit DB $\varphi(x,y,t)=x-f(y,\eta)$, and we repeat the same process to obtain (4.8) and (5.8). Finally we get

$$\frac{\partial \omega}{\partial \eta}=\frac{\partial F(\omega)}{\partial y}+\frac{\partial}{\partial y}\left[K(\omega)\frac{\partial \omega}{\partial y}\right] \tag{5.17}$$

where

$$F(\omega)\equiv -[V^1]\left[\sum_{j=1}^{N} H_j(\omega)\right]^{-1}\ .$$

ACKNOWLEDGMENTS

We are grateful to U. Frisch, N. Liron, B. Malomed, L. Pismen, Y. Pomeau, and J. Rubinstein for useful discussions. A.A.N. acknowledges the support of the Barecha Foundation.

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